Cooperative Games with Perfect Information

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Discrete time case.

In what follows as basic model we shall consider the game in extensive form with perfect information.

Definition 2. A game tree is a finite oriented treelike graph K with the root x_0. We shall use the following notations. Let x be some vertex (position). We denote by $K(x)$ a subtree K with the root in x. We denote by $Z(x)$ immediate successors of x. The vertices y, directly following after x, are called alternatives in x ($y \in Z(x)$). The player who makes a decision in x (who selects the next alternative position in x), will be denoted by $i(x)$. The choice of player $i(x)$ in position x will be denoted by $\bar{x} \in Z(x)$.

Let $N = \{1, \dots, n\}$ — be the set of all players in the game.

Definition 3. A game in extensive form with perfect information (see Kuhn (1953)) $G(x_0)$ is a graph tree $K(x_0)$, with the following additional properties:

- The set of vertices (positions) is split up into $n+1$ subsets $P_1, P_2, \dots, P_{n+1}$, which form a partition of the set of all vertices of the graph tree K. The vertices (positions) $x \in P_i$ are called players i personal positions, $i = 1, \dots, n$; vertices (positions) $x \in P_{n+1}$ are called terminal positions.
- In each final vertex (position) the system of real numbers $h(w) = (h_1(w), \dots, h_n(w))$, $w \in P_{n+1}$, $h_i(w) \geqslant 0$, $i = 1, \dots, n$ is defined. Where $h_i(w)$ is the payoff of player i in the final vertex (position).

Definition 4. A strategy of player i is a mapping $U_i(\cdot)$, which associate to each position $x \in P_i$ a unique alternative $y \in Z(x)$.
As in the previous case denote by $H_i(x; u_1(\cdot), \dots, u_n(\cdot))$ the payoff function od player $i \in N$ in the subgame $G(x)$ starting from the position x.

$$H_i(x; u_1(\cdot), \dots, u_n(\cdot)) = h_i(x'_l)$$

where $x'_l \in P_{n+1}$ is the last vertex (position) in the path $x = (x'_1, x'_2, \dots, x'_l)$ realized in the subgame $G(x)$, when the n-tuple of strategies $(u_1(\cdot), \dots, u_n(\cdot))$ is played.
Denote by $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot))$ the n-tuple of strategies and the trajectory (path) $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_m)$, $\bar{x}_m \in P_{n+1}$ such that

$$\max_{u_1(\cdot), \dots, u_n(\cdot)} \sum_{i=1}^{n} H_i(x_0; u_1(\cdot), \dots, u_n(\cdot)) =$$
$$= \sum_{i=1}^{n} H_i(x_0; \bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot)) = \sum_{i=1}^{n} h_i(\bar{x}_m). \qquad (15)$$

The path $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_m)$ satisfying Eq. (15) we shall call "optimal cooperative trajectory".
Define in $G(x_0)$ characteristic function in a classical way

$$V(x_0; N) = \sum_{i=1}^{n} h_i(\bar{x}_m),$$
$$V(x_0; \emptyset) = 0,$$
$$V(x_0; S) = Val\ \Gamma_{S, N\setminus S}(x_0),$$

where $Val\ \Gamma_{S, N\setminus S}(x_0)$ is a value of zero-sum game played between coalition S acting as first player and coalition $N \setminus S$ acting as player 2, with payoff of player S equal to

$$\sum_{i \in S} H_i(x_0; u_1(\cdot), \ldots, u_n(\cdot)).$$

Define $L(x_0)$ as imputation set in the game $G(x_0)$.

$$L(x_0) = \left\{ \alpha = (\alpha_1, \ldots, \alpha_n) : \alpha_i \geqslant V(x_0; \{i\}), \quad \sum_{i \in N} \alpha_i = V(x_0; N) \right\}.$$

Regularized game $G_\alpha(x_0)$. For every $\alpha \in L(x_0)$ define the noncooperative game $G_\alpha(x_0)$, which differs from the game $G(x_0)$ only by payoffs defined along optimal cooperative path $\bar{x} = (\bar{x}_0, \dots, \bar{x}_m)$. Let $\alpha \in L(x_0)$. Define the imputation distribution procedure (IDP) as function $\beta_k = (\beta_1(k), \dots, \beta_n(k))$, $k = 0, 1, \dots, m$ such that

$$\alpha_i = \sum_{k=0}^{m} \beta_i(k). \tag{16}$$

Define by $H_i^\alpha(x_0; u_1(\cdot), \dots, u_n(\cdot))$ the payoff function in the game $G_\alpha(x_0)$ and by $\bar{x} = \{\bar{x}_0, \dots, \bar{x}_m\}$ the cooperative path

$$H_i^\alpha(x_0; u_1(\cdot), \dots, u_n(\cdot)) = H_i(x_0; u_1(\cdot), \dots, u_n(\cdot))$$

for all $u_1(\cdot), \dots, u_n(\cdot)$ such that the path $x = \{x_0, \dots, x_m\}$ differs from $\bar{x} = \{\bar{x}_0, \dots, \bar{x}_m\}$, and

$$H_i^\alpha(x_0; \bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot)) = \alpha_i.$$

By the definition of the payoff function in the game $G_\alpha(x_0)$ we get that the payoffs along the optimal cooperative trajectory are equal to the components of the imputation $\alpha = (\alpha_1, \dots, \alpha_n)$.

Consider current subgames $G(\bar{x}_k)$ along the optimal path $\bar{x}$ and current imputation sets $L(\bar{x}_k)$. Let $\alpha^k \in L(\bar{x}_k)$.
Definition 5. The game $G_\alpha(x_0)$ is called regularization of the game $G(x_0)$ (α-regularization) if the IDP β is defined in such a way that

$$\alpha_i^k = \sum_{j=k}^{m} \beta_i(j)$$

or $\beta_i(k) = \alpha_i^k - \alpha_i^{k+1}$, $i \in N$, $k = 0, 1, \ldots, m-1$, $\beta_i(m) = \alpha_i^m$, $\alpha_i^0 = \alpha_i$.
Theorem 2. In the regularization of the game $G_\alpha(x_0)$ there exist a Nash equilibrium with payoffs $\alpha = (\alpha_1, \ldots, \alpha_n)$.
Proof. Along the cooperative path we have

$$\alpha_i^k \geqslant V(\bar{x}_k; \{i\}), \quad i \in N, k = 0, 1, \ldots, m.$$

since $\alpha^k = (\alpha_1^k, \ldots, \alpha_n^k) \in L(\bar{x}_k)$ is an imputation in $G(\bar{x}_k)$ (note that here $V(\bar{x}_k; \{i\})$ is computed in the subgame $G(\bar{x}_k)$ but not $G_\alpha(\bar{x}_k)$). In the same time

$$\alpha_i^k = \sum_{j=k}^{m} \beta_i(j)$$

and we get

$$\sum_{j=k}^{m} \beta_i(j) \geqslant V(\bar{x}_k; \{i\}), \quad i \in N, k = 0, 1, \ldots, m. \tag{17}$$

But $\sum_{j=k}^{m} \beta_i(j)$ is the payoff of player i in the subgame $G_\alpha(\bar{x}_k)$ along the cooperative path, and from (17) using the arguments similar to those in the proof of Theorem 1 one can construct the Nash equilibrium with payoffs $\alpha = (\alpha_1, \ldots, \alpha_n)$ and resulting cooperative path $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_m)$.

Example. In this example as an imputation we shall consider Shapley value [Shapley (1953)]. Using the proposed regularization of the game we shall see that there exist a Nash equilibrium with payoffs equal to the components of the Shapley value.

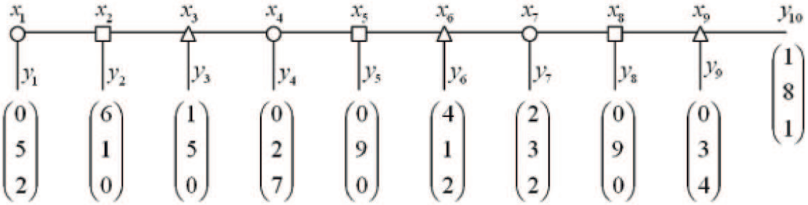


Fig. 1. Game $G(x_0)$

In the game $G(x_0)$, $N = \{1, 2, 3\}$, $P_1 = \{x_1, x_4, x_7\}$, $P_2 = \{x_2, x_5, x_8\}$, $P_3 = \{x_3, x_6, x_9\}$, $P_4 = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$. $h(y_1) = (0, 5, 2)$, $h(y_2) = (6, 1, 0)$, $h(y_3) = (1, 5, 0)$, $h(y_4) = (0, 2, 7)$, $h(y_5) = (0, 9, 0)$, $h(y_6) = (4, 1, 2)$, $h(y_7) = (2, 3, 2)$, $h(y_8) = (0, 9, 0)$, $h(y_9) = (0, 3, 4)$, $h(y_{10}) = (1, 8, 1)$. The cooperative path is $\bar{x} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6, \bar{x}_7, \bar{x}_8, \bar{x}_9, \bar{y}_{10}\}$.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_{10}
$V(x;\{1\})$	0	0	0	0	0	2	2	0	0	1
$V(x;\{2\})$	2	2	2	2	9	1	3	9	3	8
$V(x;\{3\})$	0	0	0	0	0	2	0	0	4	1
$V(x;\{1,2\})$	7	7	6	9	9	5	9	9	3	9
$V(x;\{2,3\})$	7	9	9	9	9	5	5	9	9	9
$V(x;\{1,3\})$	6	6	6	7	0	6	4	0	4	2
$V(x;\{1,2,3\})$	10	10	10	10	10	10	10	10	10	10
$Sh(x;\{1\})$	$\frac{17}{6}$	$\frac{13}{6}$	$\frac{12}{6}$	$\frac{16}{6}$	$\frac{2}{6}$	$\frac{22}{6}$	$\frac{24}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	1
$Sh(x;\{2\})$	$\frac{26}{6}$	$\frac{28}{6}$	$\frac{27}{6}$	$\frac{28}{6}$	$\frac{56}{6}$	$\frac{16}{6}$	$\frac{30}{6}$	$\frac{56}{6}$	$\frac{26}{6}$	8
$Sh(x;\{3\})$	$\frac{17}{6}$	$\frac{19}{6}$	$\frac{21}{6}$	$\frac{16}{6}$	$\frac{2}{6}$	$\frac{22}{6}$	$\frac{6}{6}$	$\frac{2}{6}$	$\frac{32}{6}$	1

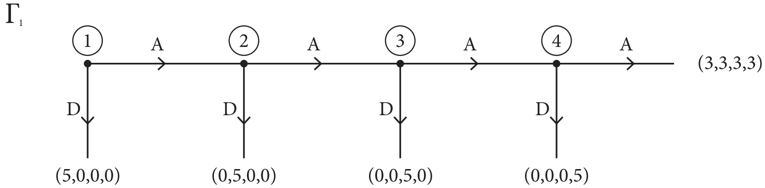
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_{10}
$\beta_1(j)$	$\frac{2}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{14}{6}$	$-\frac{28}{6}$	$-\frac{2}{6}$	$\frac{22}{6}$	0	$-\frac{4}{6}$	1
$\beta_2(j)$	$-\frac{1}{6}$	$-\frac{2}{6}$	$\frac{1}{6}$	$-\frac{28}{6}$	$\frac{40}{6}$	$-\frac{14}{6}$	$-\frac{28}{6}$	$\frac{30}{6}$	$-\frac{22}{6}$	8
$\beta_3(j)$	$-\frac{1}{6}$	$-\frac{2}{6}$	$-\frac{2}{6}$	$\frac{14}{6}$	$-\frac{20}{6}$	$\frac{16}{6}$	$\frac{4}{6}$	$-\frac{30}{6}$	$\frac{26}{6}$	1

It can be easily seen that the inequality (17)

$$\sum_{j=k}^{m} \beta_i(j) \geqslant V(\bar{x}_k; \{i\})$$

for $i \in N$ holds in this case.

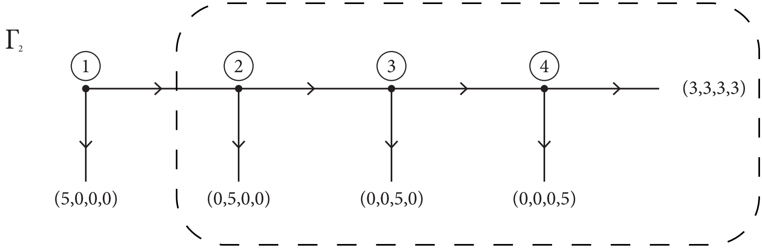
Example.



(D, D, D, D) – NE, (A, A, A, A) – NOT NE

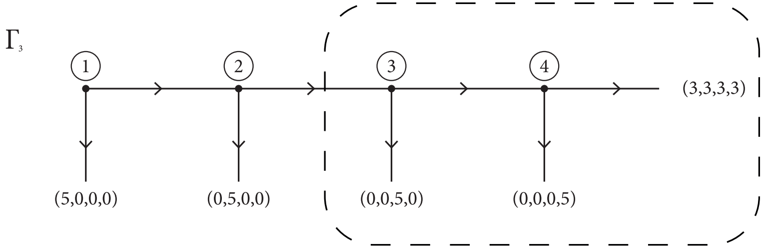
Characteristic Function of the game Γ_1 (C.f. of Γ_1)
$v_1(1,2,3,4) = 12$, $v_1(1,2,3) = 5$, $v_1(1,3,4) = 5$, $v_1(2,3,4) = 0$, $v_1(1,2,4) = 5$,
$v_1(1,2) = 5$, $v_1(1,3) = 5$, $v_1(1,4) = 5$, $v_1(2,3) = 0$, $v_1(2,4) = 0$, $v_1(3,4) = 0$,
$v_1(1) = 5$, $v_1(2) = 0$, $v_1(3) = 0$, $v_1(4) = 0$.

$Sh^1 = \left(\frac{27}{4}, \frac{7}{4}, \frac{7}{4}, \frac{7}{4}\right)$



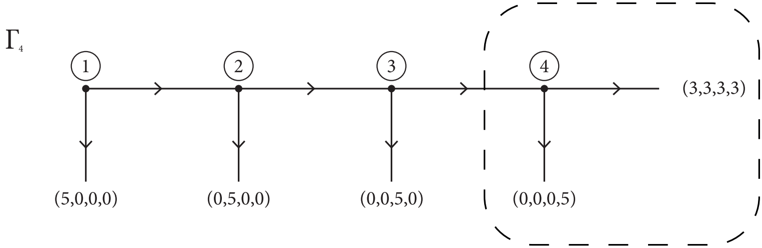
C.f. of Γ_2
$v_2(1,2,3,4) = 12,\ v_2(1,2,3) = 5,\ v_2(1,3,4) = 5,\ v_2(2,3,4) = 9,$
$v_2(1,2) = 5,\ v_2(1,3) = 0,\ v_2(1,4) = 0,\ v_2(2,3) = 5,\ v_2(2,4) = 5,\ v_2(3,4) = 0,$
$v_2(1) = 0,\ v_2(2) = 5,\ v_2(3) = 0,\ v_2(4) = 0.$

$Sh^2 = \left(\frac{19}{12}, \frac{65}{12}, \frac{30}{12}, \frac{30}{12}\right)$



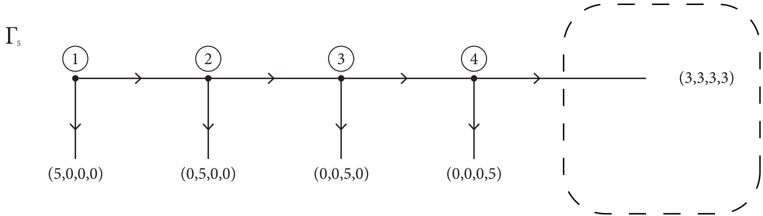
C.f. of Γ_3
$v_3(1,2,3,4) = 12,\ v_3(1,2,3) = 5,\ v_3(1,3,4) = 9,\ v_3(2,3,4) = 9,\ v_3(1,2,4) = 0$
$v_3(1,2) = 0,\ v_3(1,3) = 5,\ v_3(1,4) = 0,\ v_3(2,3) = 5,\ v_3(2,4) = 0,\ v_3(3,4) = 6,$
$v_3(1) = 0,\ v_3(2) = 0,\ v_3(3) = 5,\ v_3(4) = 0.$

$Sh^3 = \left(1, 1, \frac{90}{12}, \frac{30}{12}\right)$



C.f. of Γ_4
$v_4(1,2,3,4) = 12,\ v_4(1,2,3) = 0,\ v_4(1,3,4) = 9,\ v_4(2,3,4) = 9,\ v_4(1,2,4) = 9$
$v_4(1,2) = 0,\ v_4(1,3) = 0,\ v_4(1,4) = 5,\ v_4(2,3) = 0,\ v_4(2,4) = 5,\ v_4(3,4) = 5,$
$v_4(1) = 0,\ v_4(2) = 0,\ v_4(3) = 0,\ v_4(4) = 5.$

$Sh^4 = \left(\frac{17}{12}, \frac{17}{12}, \frac{17}{12}, \frac{93}{12}\right)$



C.f. of Γ_5
$v_5(1,2,3,4) = 12,\ v_5(1,2,3) = v_5(1,3,4) = v_5(2,3,4) = v_5(1,2,4) = 9$
$v_5(1,2) = v_5(1,3) = v_5(1,4) = v_5(2,3) = v_5(2,4) = v_5(3,4) = 6,$
$v_5(1) = v_5(2) = v_5(3) = v_5(4) = 3.$

$Sh^5 = (3,3,3,3)$

IDP (Imputation Distribution Procedure)

$\beta_k, k = 1, \ldots, 5$

$Sh^1 = \beta_1 + Sh^2, Sh^2 = \beta_2 + Sh^3, \ldots, Sh^4 = \beta_4 + Sh^5$

$\beta_1 = (Sh^1 - Sh^2), \beta_2 = (Sh^2 - Sh^3), \beta_3 = (Sh^3 - Sh^4), \beta_4 = (Sh^4 - Sh^5), \beta_5 = Sh^5$

$\sum_{k=1}^5 \beta_k = Sh^1, \sum_{k=2}^5 \beta_k = Sh^2, \sum_{k=3}^5 \beta_k = Sh^3,$
$\sum_{k=4}^5 \beta_k = Sh^4, \sum_{k=5}^5 \beta_k = Sh^5$

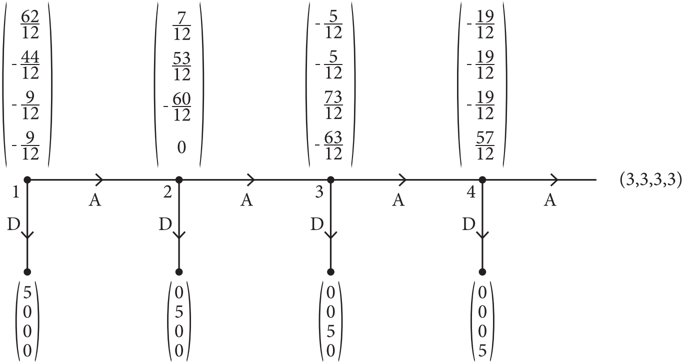
$\beta_1 = (\frac{62}{12}, -\frac{44}{12}, -\frac{9}{12}, -\frac{9}{12})$

$\beta_2 = (\frac{7}{12}, \frac{53}{12}, -\frac{60}{12}, 0)$

$\beta_3 = (-\frac{5}{12}, -\frac{5}{12}, \frac{73}{12}, -\frac{63}{12})$

$\beta_4 = (-\frac{19}{12}, -\frac{19}{12}, -\frac{19}{12}, \frac{57}{12})$

$\beta_5 = (3, 3, 3, 3)$



(A, A, A, A) – NE

$$NE \begin{cases} \frac{62}{12} + \frac{7}{12} - \frac{5}{12} - \frac{19}{12} + 3 > 5 \\ \frac{53}{12} - \frac{5}{12} - \frac{19}{12} + 3 > 5 \\ \frac{73}{12} - \frac{19}{12} + 3 > 5 \\ \frac{57}{12} + 3 > 5 \end{cases}$$

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